

# FUNCTORIAL CARTIER DUALITY

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## INTRODUCTION

In this paper we obtain the Cartier duality for  $k$ -schemes of commutative monoids functorially without providing the vector spaces of functions with a topology (as in [DGr, Exposé VII<sub>B</sub> by P. Gabriel, 2.2.1]), generalizing a result for finite commutative algebraic groups by M. Demazure & P. Gabriel ([DG, II, §1, 2.10]).

All functors we consider are functors defined over the category of commutative  $k$ -algebras. Given a  $k$ -module  $E$ , we denote by  $\mathbf{E}$  the functor of  $k$ -modules  $\mathbf{E}(B) := E \otimes_k B$ . Given two functors of  $k$ -modules  $F$  and  $H$ ,  $\mathbf{Hom}_k(F, H)$  will denote the functor of  $k$ -modules

$$\mathbf{Hom}_k(F, H)(B) = \mathrm{Hom}_B(F|_B, H|_B)$$

where  $F|_B$  is the functor  $F$  restricted to the category of commutative  $B$ -algebras. We denote  $F^* := \mathbf{Hom}_k(F, \mathbf{k})$ . It holds that  $\mathbf{E}^{**} = \mathbf{E}$  ([A1, 1.10]). Given  $\mathbf{E}$  there exists a  $k$ -module  $V$  such that  $\mathbf{E} = \mathbf{V}^*$  if and only if  $E$  is a projective  $k$ -module of finite type.

Given a functor of commutative  $k$ -algebras  $\mathcal{A}$ , we define  $\mathrm{Spec} \mathcal{A}$  to be the functor  $\mathrm{Spec} \mathcal{A}(B) := \mathrm{Hom}_{k\text{-alg}}(\mathcal{A}, B)$ . Let  $X = \mathrm{Spec} \mathcal{A}$  be a  $k$ -scheme and let  $X^\cdot$  be its functor of points ( $X^\cdot(B) := \mathrm{Hom}_{k\text{-alg}}(\mathcal{A}, B)$ ). It is easy to see that  $X^\cdot = \mathrm{Spec} \mathbf{A}$ . If  $\mathbf{C}^*$  is a functor of commutative algebras, we prove (Theorem 1.6) that  $\mathrm{Spec} \mathbf{C}^*$  is a direct limit (of functor of points) of finite  $k$ -schemes.

If  $\mathcal{A} = \mathbf{A}$ , in [A1, 3.4] we proved that the linear enveloping of  $\mathrm{Spec} \mathcal{A}$  is  $\mathcal{A}^*$ , that is,

$$\mathrm{Hom}_{\mathrm{functors}}(\mathrm{Spec} \mathcal{A}, H) = \mathrm{Hom}_k(\mathcal{A}^*, H)$$

for all dual functors of  $k$ -modules  $H$  (i.e.,  $H = F^*$ ). If in addition  $G = \mathrm{Spec} \mathcal{A}$  is an affine  $k$ -group, in [A1, 5.4] we proved that the algebraic enveloping of  $\mathrm{Spec} \mathcal{A}$  is  $\mathcal{A}^*$  again, that is,

$$\mathbf{Hom}_{\mathrm{monoids}}(\mathrm{Spec} \mathcal{A}, \mathcal{B}) = \mathbf{Hom}_{k\text{-alg}}(\mathcal{A}^*, \mathcal{B})$$

for all dual functors of  $k$ -algebras  $\mathcal{B}$ . As a consequence, we obtained that the category of  $G$ -modules is equivalent to the category of  $\mathcal{A}^*$ -modules.

In this paper we prove simultaneously the same results for  $\mathcal{A} = \mathbf{A}$ ,  $\mathbf{C}^*$  (Corollary 2.4, Proposition 2.7, Theorem 2.8). In particular,

$$\mathbf{Hom}_{\mathrm{functors}}(\mathrm{Spec} \mathcal{A}, \mathbf{k}) = \mathbf{Hom}_k(\mathcal{A}^*, \mathbf{k}) = \mathcal{A}.$$

Moreover, if  $\mathrm{Spec} \mathcal{A}$  is a functor of abelian monoids, then we get

$$\mathbf{Hom}_{\mathrm{monoids}}(\mathrm{Spec} \mathcal{A}, \mathbf{k}) = \mathbf{Hom}_{k\text{-alg}}(\mathcal{A}^*, \mathbf{k}) = \mathrm{Spec} \mathcal{A}^*$$

where  $\mathbf{k}$  is regarded as a functor of monoids with the multiplication operation.

Given a functor of monoids  $H$ , we say that  $H^* = \mathbf{Hom}_{\mathrm{monoids}}(H, \mathbf{k})$  is the dual functor of monoids of  $H$  and we prove the following theorem.

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*Date:* June, 2007.

*2000 Mathematics Subject Classification.* Primary 14L15. Secondary 20M07, 22D35.

**Theorem 1.** *The category of abelian affine monoids  $G = \text{Spec } A$  is anti-equivalent to the category of functors  $\text{Spec } \mathbf{C}^*$  of abelian monoids. The functors giving the anti-equivalence are the ones that assign to each functor of monoids its dual functor of monoids.*

In particular,  $G^{**} = G$ .

Finally, we get three immediate applications of the functorial Cartier duality:

- (1) The tangent space to  $G^*$  at the identity element is isomorphic to the vector space  $V$  of linear functions of  $G$ .
- (2) If  $\text{car } k = 0$  and  $G$  is a unipotent abelian  $k$ -group, then it is isomorphic to the dual of  $V$ .
- (3) If  $G$  is semisimple, then the transposed Fourier transform of  $G^*$  is the inverse of the Fourier transform of  $G$ .

## 1. SPECTRUM OF A FUNCTOR OF ALGEBRAS

The basic references for reading this paper are [A1] and [S].

Let  $k$  be a commutative ring with unit. If  $E$  is a  $k$ -module we will say that  $\mathbf{E}$  is a cuasi-coherent  $k$ -module. The category of  $k$ -modules is equivalent to the category of cuasi-coherent  $k$ -modules ([A1, 1.12]). In particular,  $\text{Hom}_k(\mathbf{E}, \mathbf{E}') = \text{Hom}_k(E, E')$ .  $\mathbf{E}^* = \mathbf{Hom}_k(\mathbf{E}, \mathbf{k})$  is the functor of points of the scheme  $\text{Spec } S_k^* E$  and we say that it is a  $k$ -module scheme. As it holds that  $\mathbf{E}^{**} = \mathbf{E}$ , the category of cuasi-coherent modules is anti-equivalent to the category of module schemes ([A1, 1.12]).

**Notation 1.1.** *Throughout this paper we assume that  $\mathbf{A}^*$  is a commutative  $k$ -algebra scheme (equivalently,  $A$  is a cocommutative  $k$ -coalgebra, [A1, 4.2]). From Proposition 1.5 on, we will always assume that  $A$  is a projective  $k$ -module.*

**Definition 1.2.** *Given a functor of  $k$ -algebras  $\mathcal{A}$ , the functor  $\text{Spec } \mathcal{A}$ , “spectrum of  $\mathcal{A}$ ”, is defined to be*

$$(\text{Spec } \mathcal{A})(B) := \text{Hom}_{k\text{-alg}}(\mathcal{A}, \mathbf{B})$$

for each commutative  $k$ -algebra  $B$ .

**Example 1.3.** *If  $C$  is a commutative  $k$ -algebra, then*

$$\text{Spec } \mathbf{C} = (\text{Spec } C)^*.$$

**Example 1.4.** *Let  $X$  be a set. Let us consider the discrete topology on  $X$ . Let  $\mathcal{X}$  be the functor, which we will call the constant functor  $X$ , defined by*

$$\mathcal{X}(B) := \text{Aplic}_{\text{cont.}}(\text{Spec } B, X)$$

for each commutative  $k$ -algebra  $B$ .

Let  $\mathcal{A}_X$  be the functor of algebras defined by

$$\mathcal{A}_X(B) := \text{Aplic}(X, B) = \prod_X B$$

for each commutative  $k$ -algebra  $B$ . Let us observe that  $\mathcal{A}_X = \prod_X \mathbf{k} = (\bigoplus_X \mathbf{k})^*$  is a commutative algebra scheme.

Let us prove that  $\text{Spec } \mathcal{A}_X = \mathcal{X}$ .

The morphisms of functors of  $k$ -algebras are, in particular, morphisms of functors of  $k$ -modules

$$(\text{Spec } \mathcal{A}_X)(B) = \text{Hom}_{k\text{-alg}}(\mathcal{A}_X, \mathbf{B}) \subset \text{Hom}_k(\mathcal{A}_X, \mathbf{B}) = \text{Hom}_k\left(\prod_{x \in X} \mathbf{k}, \mathbf{B}\right)$$

and from [A1, 4.5] we know that morphisms of functors of  $k$ -modules from  $\prod_X \mathbf{k}$  to  $\mathbf{B}$  factorize through the projection onto a finite number of factors. Therefore, given

a morphism  $\prod_X \mathbf{k} \rightarrow \mathbf{B}$  there exists a subset  $Y \subset X$  of finite order, such that we get the factorization

$$\prod_X \mathbf{k} \rightarrow \prod_Y \mathbf{k} \rightarrow \mathbf{B}.$$

Every continuous morphism  $\text{Spec } B \rightarrow X$  has finite image, because  $\text{Spec } B$  is compact and  $X$  is discrete. We can assume that  $X$  is a finite set.

Thus, we get

$$\begin{aligned} \text{Hom}_{k\text{-alg}}(\prod_X \mathbf{k}, \mathbf{B}) &= \text{Hom}_{k\text{-alg}}(\prod_X k, B) \\ &= \text{Cont.Aplic}_{\text{Spec } k}(\text{Spec } B, \text{Spec } \prod_X k = \prod_X \text{Spec } k) \\ &= \text{Aplic}_{\text{cont.}}(\text{Spec } B, X). \end{aligned}$$

Later we will see that the ring of functions of  $\text{Spec } \mathcal{A}_X$  coincides with  $\mathcal{A}_X$ , that is,

$$\mathbf{Hom}_{\text{functors}}(\text{Spec } \mathcal{A}_X, \mathbf{k}) = \mathcal{A}_X.$$

**Proposition 1.5.** *Let  $\mathcal{A}$  a functor of commutative algebras. Then,  $\text{Spec } \mathcal{A} = \mathbf{Hom}_{k\text{-alg}}(\mathcal{A}, \mathbf{k})$ .*

*Proof.* By the adjoint functor formula ([A1, 1.15]) (restricted to the morphisms of algebras) it holds that

$$\begin{aligned} \mathbf{Hom}_{k\text{-alg}}(\mathcal{A}, \mathbf{k})(B) &= \text{Hom}_{B\text{-alg}}(\mathcal{A}|_B, \mathbf{B}) = \text{Hom}_{k\text{-alg}}(\mathcal{A}, \mathbf{B}) \\ &= (\text{Spec } \mathcal{A})(B). \end{aligned}$$

□

Therefore,  $\text{Spec } \mathbf{A}^* = \mathbf{Hom}_{k\text{-alg}}(\mathbf{A}^*, \mathbf{k}) \subset \mathbf{Hom}_k(\mathbf{A}^*, \mathbf{k}) = \mathbf{A}$ .

If  $E$  is a finitely-generated  $k$ -module, we will say that  $\mathbf{E}$  is coherent.

**Theorem 1.6.** *Let  $\mathbf{A}^*$  be a commutative algebra scheme and let us assume that  $A$  is a projective  $k$ -module. By [A1, 4.12],  $\mathbf{A}^* = \varprojlim_i \mathbf{A}_i$ , where  $\mathbf{A}_i$  are the coherent algebras that are quotients of  $\mathbf{A}^*$ . It holds that*

$$\text{Spec } \mathbf{A}^* = \varinjlim_i \text{Spec } \mathbf{A}_i.$$

*Proof.* By [A1, 4.5], the image of every  $k$ -linear morphism  $\mathbf{A}^* \rightarrow \mathbf{B}$  is a coherent  $k$ -module; therefore, every morphism of functors of  $k$ -algebras  $\mathbf{A}^* \rightarrow \mathbf{B}$  factorizes through a coherent algebra  $\mathbf{A}_i$  that is a quotient of  $\mathbf{A}^*$ . Then,

$$\begin{aligned} (\text{Spec } \mathbf{A}^*)(B) &= \text{Hom}_{k\text{-alg}}(\mathbf{A}^*, \mathbf{B}) = \varinjlim_i \text{Hom}_{k\text{-alg}}(\mathbf{A}_i, \mathbf{B}) \\ &= \varinjlim_i (\text{Spec } \mathbf{A}_i)(B). \end{aligned}$$

□

**Proposition 1.7.** *Let  $\mathcal{B}$  be a functor of commutative  $k$ -algebras. Then,*

- (1)  $\text{Hom}_{\text{functors}}(\text{Spec } \mathbf{A}, \text{Spec } \mathcal{B}) = \text{Hom}_{k\text{-alg}}(\mathcal{B}, \mathbf{A})$ .
- (2)  $\text{Hom}_{\text{functors}}(\text{Spec } \mathbf{A}^*, \text{Spec } \mathcal{B}) = \text{Hom}_{k\text{-alg}}(\mathcal{B}, \mathbf{A}^*)$ .

*Proof.* It holds that  $\text{Hom}_{\text{functors}}((\text{Spec } A)^\cdot, F) = F(A)$  for every functor  $F$ , by Yoneda's lemma.

(1)

$$\begin{aligned}\mathrm{Hom}_{\mathbf{functors}}(\mathrm{Spec} \mathbf{A}, \mathrm{Spec} \mathcal{B}) &= \mathrm{Hom}_{\mathbf{functors}}((\mathrm{Spec} \mathbf{A})^\cdot, \mathrm{Spec} \mathcal{B}) \\ &= \mathrm{Hom}_{k\text{-alg}}(\mathcal{B}, \mathbf{A}).\end{aligned}$$

(2)

$$\begin{aligned}\mathrm{Hom}_{\mathbf{functors}}(\mathrm{Spec} \mathbf{A}^*, \mathrm{Spec} \mathcal{B}) &= \mathrm{Hom}_{\mathbf{functors}}(\varinjlim_i \mathrm{Spec} \mathbf{A}_i, \mathrm{Spec} \mathcal{B}) \\ &= \varprojlim_i \mathrm{Hom}_{\mathbf{functors}}(\mathrm{Spec} \mathbf{A}_i, \mathrm{Spec} \mathcal{B}) \\ &= \varprojlim_i \mathrm{Hom}_{k\text{-alg}}(\mathcal{B}, \mathbf{A}_i) \\ &= \mathrm{Hom}_{k\text{-alg}}(\mathcal{B}, \mathbf{A}^*).\end{aligned}$$

□

**Proposition 1.8.** *For  $\mathcal{A} = \mathbf{A}, \mathbf{A}^*$ , it holds that*

$$\mathbf{Hom}_{\mathbf{functors}}(\mathrm{Spec} \mathcal{A}, \mathbf{k}) = \mathcal{A}.$$

*Proof.*

$$\begin{aligned}\mathbf{Hom}_{\mathbf{functors}}(\mathrm{Spec} \mathcal{A}, \mathbf{k}) &= \mathbf{Hom}_{\mathbf{functors}}(\mathrm{Spec} \mathcal{A}, \mathrm{Spec} \mathbf{k}[\mathbf{x}]) \\ &= \mathbf{Hom}_{k\text{-alg}}(\mathbf{k}[\mathbf{x}], \mathcal{A}) = \mathcal{A}.\end{aligned}$$

□

Explicitly,  $\mathcal{A} = \mathbf{Hom}_{\mathbf{functors}}(\mathrm{Spec} \mathcal{A}, \mathbf{k})$ ,  $w \mapsto \tilde{w}$ , where  $\tilde{w}(\phi) = \phi(w)$ , for every  $\phi \in \mathrm{Spec} \mathcal{A} = \mathbf{Hom}_{k\text{-alg}}(\mathcal{A}, \mathbf{k})$ .

## 2. CLOSURE OF DUAL MODULES AND ALGEBRAS

**Definition 2.1.** *We will say that a functor of  $k$ -modules  $F$  is dual if there exists a functor of  $k$ -modules  $H$  such that  $F \simeq H^*$ .*

**Proposition 2.2.** *Let  $F$  be a functor of  $k$ -modules such that  $F^*$  is a reflexive functor. The closure of dual functors of  $k$ -modules of  $F$  is  $F^{**}$ , that is, it holds the functorial equality*

$$\mathrm{Hom}_k(F, G) = \mathrm{Hom}_k(F^{**}, G)$$

*for every dual functor of  $k$ -modules  $G$ .*

*Proof.* Let us write  $G = H^*$ , then

$$\begin{aligned}\mathrm{Hom}_k(F, G) &= \mathrm{Hom}_k(F, H^*) = \mathrm{Hom}_k(H, F^*) = \mathrm{Hom}_k(H, F^{***}) \\ &= \mathrm{Hom}_k(F^{**}, H^*) = \mathrm{Hom}_k(F^{**}, G).\end{aligned}$$

□

**Definition 2.3.** *Let  $F$  be a functor of sets. We will denote by  $k[F]$  the functor over the category of commutative  $k$ -algebras defined by  $k[F](B) = \{\text{formal finite } B\text{-linear combinations of } f \in F(B)\}$ .*

**Corollary 2.4.** *Let  $\mathcal{A}$  be a functor of commutative  $k$ -algebras. Let us assume that  $\mathbf{Hom}_{\mathbf{functors}}(\mathrm{Spec} \mathcal{A}, \mathbf{k}) = \mathcal{A}$  and that  $\mathcal{A}$  is a reflexive functor of  $k$ -modules. Then, the closure of dual functors of  $k$ -modules of  $k[\mathrm{Spec} \mathcal{A}]$  is  $\mathcal{A}^*$ , that is,*

$$\mathrm{Hom}_{\mathbf{functors}}(\mathrm{Spec} \mathcal{A}, F) = \mathrm{Hom}_k(k[\mathrm{Spec} \mathcal{A}], F) = \mathrm{Hom}_k(\mathcal{A}^*, F)$$

*for every dual functor of  $k$ -modules  $F$ .*

*Proof.* As  $k[\mathrm{Spec} \mathcal{A}]^* = \mathbf{Hom}_{\mathbf{functors}}(\mathrm{Spec} \mathcal{A}, \mathbf{k}) = \mathcal{A}$ , the closure of dual functors of  $k$ -modules of  $k[\mathrm{Spec} \mathcal{A}]$  is  $k[\mathrm{Spec} \mathcal{A}]^{**} = \mathcal{A}^*$ . □

**Proposition 2.5.** *Let  $F$  be a functor of  $k$ -algebras such that  $F^*$  is a reflexive functor of  $k$ -modules. The closure of dual functors of  $k$ -algebras of  $F$  is  $F^{**}$ , that is, it holds the functorial equality*

$$\mathrm{Hom}_{k\text{-alg}}(F, G) = \mathrm{Hom}_{k\text{-alg}}(F^{**}, G)$$

for every dual functor of  $k$ -algebras  $G$ . In particular,  $\mathrm{Spec} F = \mathrm{Spec} F^{**}$ .

*Proof.* Let us observe that

$$\begin{aligned} (F \otimes .^n \otimes F)^* &= \mathrm{Hom}_k(F \otimes .^n \otimes F, \mathbf{k}) = \mathrm{Hom}_k(F, (F \otimes .^{n-1} \otimes F)^*) \\ &\stackrel{\text{Induction}}{=} \mathrm{Hom}_k(F, (F^{**} \otimes .^{n-1} \otimes F^{**})^*) = \mathrm{Hom}_k(F^{**}, (F^{**} \otimes .^{n-1} \otimes F^{**})^*) \\ &= (F^{**} \otimes .^n \otimes F^{**})^*. \end{aligned}$$

Therefore, given a dual functor of  $k$ -modules  $S^*$ ,

$$\begin{aligned} \mathrm{Hom}_k(F \otimes \dots \otimes F, S^*) &= \mathrm{Hom}_k(S, (F \otimes \dots \otimes F)^*) \\ &= \mathrm{Hom}_k(S, (F^{**} \otimes \dots \otimes F^{**})^*) \\ &= \mathrm{Hom}_k(F^{**} \otimes \dots \otimes F^{**}, S^*). \end{aligned}$$

If we consider  $S^* = F^{**}$ , it follows easily that the structure of algebra of  $F$  defines a structure of algebra on  $F^{**}$ . Finally, if we consider  $S^* = G$ , we obtain that  $\mathrm{Hom}_{k\text{-alg}}(F, G) = \mathrm{Hom}_{k\text{-alg}}(F^{**}, G)$ .  $\square$

**Remark 2.6.** *Let us observe that if  $F$  is also a functor of commutative algebras, then so is  $F^{**}$ : since  $\mathrm{Hom}_k(F \otimes F, F^{**}) = \mathrm{Hom}_k(F^{**} \otimes F^{**}, F^{**})$ , the morphism  $F \otimes F \rightarrow F$ ,  $f \otimes f' \mapsto ff' - f'f = 0$  extends to a unique morphism  $F^{**} \otimes F^{**} \rightarrow F^{**}$  (which is  $f \otimes f' \mapsto ff' - f'f = 0$ ).*

Let  $\mathbf{A}^*$  and  $\mathbf{B}^*$  be commutative  $k$ -algebra schemes. By [A1, 1.8], we have that  $(\mathbf{A}^* \otimes_k \mathbf{B}^*)^* = \mathbf{Hom}_k(\mathbf{A}^* \otimes_k \mathbf{B}^*, \mathbf{k}) = \mathbf{Hom}_k(\mathbf{A}^*, \mathbf{B}) = \mathbf{A} \otimes_k \mathbf{B}$ . Then,  $\mathrm{Spec} \mathbf{A}^* \times \mathrm{Spec} \mathbf{B}^* = \mathrm{Spec}(\mathbf{A}^* \otimes_k \mathbf{B}^*) \stackrel{2.5}{=} \mathrm{Spec}(\mathbf{A} \otimes_k \mathbf{B})^*$ .

Let  $\mathcal{A}$  be a functor of  $k$ -algebras and let us assume that  $G = \mathrm{Spec} \mathcal{A}$  is a functor of monoids. The functor of  $k$ -modules  $k[G]$  is obviously a functor of  $k$ -algebras. Given a functor of  $k$ -algebras  $\mathcal{B}$ , it is easy to check the equality

$$\mathrm{Hom}_{\text{monoids}}(G, \mathcal{B}) = \mathrm{Hom}_{k\text{-alg}}(k[G], \mathcal{B}).$$

**Proposition 2.7.** *Let  $G = \mathrm{Spec} \mathcal{A}$  be a functor of monoids. Let us assume that  $\mathbf{Hom}_{\text{functors}}(G, \mathbf{k}) = \mathcal{A}$  and that  $\mathcal{A}$  is a reflexive functor of  $k$ -modules. Then the closure of dual functors of algebras of  $k[G]$  coincides with  $\mathcal{A}^*$ , that is,*

$$\mathrm{Hom}_{\text{monoids}}(G, \mathcal{B}) = \mathrm{Hom}_{k\text{-alg}}(k[G], \mathcal{B}) = \mathrm{Hom}_{k\text{-alg}}(\mathcal{A}^*, \mathcal{B})$$

for every dual functor of  $k$ -algebras  $\mathcal{B}$ .

*Proof.*  $k[G]^* = \mathcal{A}$  is reflexive, then the closure of dual functors of algebras of  $k[G]$  is  $\mathcal{A}^*$ .  $\square$

**Theorem 2.8.** *Let  $G = \mathrm{Spec} \mathcal{A}$  be a functor of monoids. Let us assume that  $\mathbf{Hom}_{\text{functors}}(\mathrm{Spec} \mathcal{A}, \mathbf{k}) = \mathcal{A}$  and that  $\mathcal{A}$  is a reflexive functor of  $k$ -modules. The category of  $G$ -modules is equivalent to the category of  $\mathcal{A}^*$ -modules. Likewise, the category of dual functors of  $G$ -modules is equivalent to the category of dual functors of  $\mathcal{A}^*$ -modules.*

*Proof.* Let  $E$  be a  $k$ -module. Let us observe that  $\mathbf{End}_k(\mathbf{E}) = (\mathbf{E}^* \otimes \mathbf{E})^*$  is a dual functor. Therefore,

$$\mathrm{Hom}_{k\text{-alg}}(k[G], \mathbf{End}_k(\mathbf{E})) = \mathrm{Hom}_{k\text{-alg}}(\mathcal{A}^*, \mathbf{End}_k(\mathbf{E})).$$

In conclusion, endowing  $E$  with a structure of  $G$ -module is equivalent to endowing  $E$  with a structure of  $\mathcal{A}^*$ -module.

Let us observe that  $\text{Hom}_k(k[G], \mathbf{E}) = \text{Hom}_k(\mathbf{E}^*, \mathcal{A}) = \text{Hom}_k(\mathcal{A}^*, \mathbf{E})$ . Hence, given any two  $G$ -modules (or  $\mathcal{A}^*$ -modules)  $E, E'$ , a linear morphism  $f : E \rightarrow E'$  and  $e \in E$ , we will have that the morphism  $f_1 : k[G] \rightarrow \mathbf{E}'$ ,  $f_1(g) := f(ge) - gf(e)$  is null if and only if the morphism  $f_2 : \mathcal{A}^* \rightarrow \mathbf{E}'$ ,  $f_2(a) := f(ae) - af(e)$  is null. In conclusion,  $\text{Hom}_{G\text{-mod}}(\mathbf{E}, \mathbf{E}') = \text{Hom}_{\mathcal{A}^*}(\mathbf{E}, \mathbf{E}')$ .  $\square$

Let us highlight that the structure of functor of algebras of  $\mathcal{A}^*$  is the one that makes the embedding  $\text{Spec } \mathcal{A} \hookrightarrow \mathcal{A}^*$  be a morphism of monoids.

**Example 2.9.** *The  $\mathbb{C}$ -linear representations of  $\mathbb{Z}$  are equivalent to the  $\mathbb{C}[\mathbb{Z}]$ -modules.  $\mathbb{C}[\mathbb{Z}] = \mathbb{C}[x, 1/x]$ ,  $n \mapsto x^n$ , is a principal ideal domain. Thus, if  $E$  is a finite  $\mathbb{C}$ -linear representation of  $\mathbb{Z}$ , then*

$$E = \bigoplus_{\alpha \neq 0, n_\alpha > 0} \mathbb{C}[x]/(x - \alpha)^{n_\alpha}$$

such that  $n \cdot (\bar{p}_{n_\alpha})_{n_\alpha} = (\overline{x^n \cdot p_{n_\alpha}})_{n_\alpha}$ .

**Proposition 2.10.** *Let  $G = \text{Spec } \mathcal{A}$  be a functor of monoids. Let us assume that  $\mathbf{Hom}_{\text{functors}}(\text{Spec } \mathcal{A}, \mathbf{k}) = \mathcal{A}$  and that  $\mathcal{A}$  is a reflexive functor of  $k$ -modules. Let  $\mathcal{V}$  be a dual functor of  $G$ -modules (on the left and on the right). Then,*

$$\text{Der}_k(\mathcal{A}^*, \mathcal{V}) = \text{Der}(G, \mathcal{V})$$

where  $\text{Der}(G, \mathcal{V}) := \{D \in \text{Hom}_{\text{functors}}(G, \mathcal{V}) : D(g \cdot g') = D(g) \cdot g' + g \cdot D(g')\}$ .

*Proof.*

$$\begin{aligned} \text{Der}_k(\mathcal{A}^*, \mathcal{V}) &= \{f \in \text{Hom}_{k\text{-alg}}(\mathcal{A}^*, \mathcal{A}^* \oplus \mathcal{V}\epsilon), \epsilon^2 = 0, f(a) = a \bmod(\epsilon)\} \\ &= \{f \in \text{Hom}_{\text{monoids}}(G, \mathcal{A}^* \oplus \mathcal{V}\epsilon), \epsilon^2 = 0, f(g) = g \bmod(\epsilon)\} \\ &= \text{Der}(G, \mathcal{V}). \end{aligned}$$

$\square$

### 3. FUNCTORIAL CARTIER DUALITY

**Definition 3.1.** *Given a functor of abelian monoids  $G$ , where we regard  $\mathbf{k}$  as a monoid with the multiplication operation,*

$$G^* := \mathbf{Hom}_{\text{monoids}}(G, \mathbf{k})$$

*will be referred to as the dual monoid of  $G$ .*

If  $G$  is a functor of groups, then  $G^* = \mathbf{Hom}_{\text{groups}}(G, G_m)$ .

**Theorem 3.2.** *Assume that  $\text{Spec } \mathcal{A}$  is a functor of abelian monoids,  $\mathcal{A}$  is a reflexive functor of  $k$ -modules and  $\mathbf{Hom}_{\text{functors}}(\text{Spec } \mathcal{A}, \mathbf{k}) = \mathcal{A}$ . Then*

$$\text{Spec } \mathcal{A}^* = (\text{Spec } \mathcal{A})^*$$

*(in particular, this equality shows that  $\text{Spec } \mathcal{A}^*$  is a functor of abelian monoids).*

*Proof.*

$$\begin{aligned} \text{Spec } \mathcal{A}^* &= \mathbf{Hom}_{k\text{-alg}}(\mathcal{A}^*, \mathbf{k}) = \mathbf{Hom}_{\text{monoids}}(\text{Spec } \mathcal{A}, \mathbf{k}) \\ &= (\text{Spec } \mathcal{A})^*. \end{aligned}$$

$\square$

**Remark 3.3.** Explicitly,  $\text{Spec } \mathcal{A}^* = \mathbf{Hom}_{\text{monoids}}(\text{Spec } \mathcal{A}, G_m)$ ,  $\phi \mapsto \tilde{\phi}$ , where  $\tilde{\phi}(x) = \phi(x)$ , for every  $\phi \in \text{Spec } \mathcal{A}^* = \mathbf{Hom}_{k\text{-alg}}(\mathcal{A}^*, k)$  and  $x \in \text{Spec } \mathcal{A} \subset \mathcal{A}^*$ .

Let  $G = \text{Spec } \mathcal{A}$ .  $G^*$  is a functor of abelian monoids  $((f \cdot f')(g) := f(g) \cdot f'(g)$ , for every  $f, f' \in G^*$  and  $g \in G$ ), the inclusion  $G^* = \mathbf{Hom}_{\text{monoids}}(G, k) \subset \mathbf{Hom}_{\text{functors}}(G, k) = \mathcal{A}$  is a morphism of monoids and the diagram

$$\begin{array}{ccc} (\text{Spec } \mathcal{A})^* & \xrightarrow{\hookrightarrow} & \mathbf{Hom}_{\text{functors}}(\text{Spec } \mathcal{A}, k) = \mathcal{A} \\ \parallel & & \parallel \\ \text{Spec } \mathcal{A}^* & \xrightarrow{\hookrightarrow} & \mathbf{Hom}_k(\mathcal{A}^*, k) = \mathcal{A}^{**} \end{array}$$

is commutative.

**Theorem 3.4.** Assume that  $G = \text{Spec } \mathcal{A}$  is a functor of abelian monoids,  $\mathcal{A}$  is a reflexive functor of  $k$ -modules and  $\mathbf{Hom}_{\text{functors}}(\text{Spec } \mathcal{A}, k) = \mathcal{A}$ ,  $\mathbf{Hom}_{\text{functors}}(\text{Spec } \mathcal{A}^*, k) = \mathcal{A}^*$ . Then:

- (1) The morphism  $G \xrightarrow{**} G^{**}$ ,  $g \mapsto g^{**}$ , where  $g^{**}(f) := f(g)$  for every  $f \in G^*$ , is an isomorphism.
- (2)  $\mathbf{Hom}_{\text{monoids}}(G_1, G_2) = \mathbf{Hom}_{\text{monoids}}(G_2^*, G_1^*)$ .

*Proof.*

- (1) It is easy to check that the diagram

$$\begin{array}{ccccc} \text{Spec } \mathcal{A}^{**} & \xrightarrow[\sim]{3.2} & (\text{Spec } \mathcal{A}^*)^* & \xleftarrow[\sim]{3.2} & (\text{Spec } \mathcal{A})^{**} \\ & \searrow & & \nearrow & \\ & & \text{Spec } \mathcal{A} & & \end{array}$$

is commutative. Hence the morphism  $**$  is an isomorphism.

- (2) Every morphism of monoids  $G_1 \rightarrow G_2$ , taking  $\mathbf{Hom}_{\text{monoids}}(-, k)$ , defines a morphism  $G_2^* \rightarrow G_1^*$ . Taking  $\mathbf{Hom}_{\text{monoids}}(-, k)$  we get the original morphism  $G_1 \rightarrow G_2$ , as it is easy to check. Likewise, every morphism of monoids  $G_2^* \rightarrow G_1^*$ , taking  $\mathbf{Hom}_{\text{monoids}}(-, k)$ , defines a morphism  $G_1 \rightarrow G_2$ . Taking  $\mathbf{Hom}_{\text{monoids}}(-, k)$  we get the original morphism  $G_2^* \rightarrow G_1^*$ .

□

**Theorem 3.5.** The category of abelian affine  $k$ -monoids  $G = \text{Spec } A$  is anti-equivalent to the category of functors  $\text{Spec } \mathbf{A}^*$  of abelian monoids (we assume the  $k$ -modules  $A$  are projective).

In particular, we get the Cartier duality for finite commutative algebraic groups ([S, §9.9]). Dieudonné ([D, Ch. I, §2, 13]) proves the equivalence between the category of  $k$ -coalgebras in groups and the dual category of linearly compact  $k$ -algebras in cogroups (where  $k$  is a field).

**Proposition 3.6.** Let  $G_1$  and  $G_2$  be a pair of functors of abelian monoids. It holds that

$$(G_1 \times G_2)^* = G_1^* \times G_2^*.$$

*Proof.*

$$\begin{aligned} (G_1 \times G_2)^* &= \mathbf{Hom}_{\text{monoids}}(G_1 \times G_2, k) \\ &= \mathbf{Hom}_{\text{monoids}}(G_1, k) \times \mathbf{Hom}_{\text{monoids}}(G_2, k) = G_1^* \times G_2^*. \end{aligned}$$

□

Let us show some examples.

**Example 3.7.** Let  $k$  be a field. Let  $\mathbb{Z} := \operatorname{Spec} \prod_{\mathbb{Z}} k$ . Obviously  $\mathbb{Z}^* = G_m$ , therefore  $G_m^* = \mathbb{Z}$ . In other words,  $\mathbf{Hom}_{\text{grupos}}(G_m, G_m) = \mathbb{Z}$ : given  $\tau : G_m \rightarrow G_m$ , there exists  $n \in \mathbb{Z}$  such that  $\tau(\alpha) = \alpha^n$ .

Given a quasi-coherent  $\prod_X k$ -module  $\mathbf{E}$ , then  $\mathbf{E} = \bigoplus_{x \in X} \mathbf{E}_x$  in the obvious way. Given a morphism of algebra schemes  $\prod_X k \rightarrow \mathbf{A}^*$ , then  $\mathbf{A}^* = \prod_{x \in X} \mathbf{A}_x^*$ . Moreover, if the morphism  $\operatorname{Spec} \mathbf{A}^* \rightarrow \operatorname{Spec} \prod_X k$  is injective, then  $\mathbf{A}_x^* = k$  or  $\mathbf{A}_x^* = 0$  for each  $x \in X$ . Therefore,  $\operatorname{Spec} \mathbf{A}^* \subset \operatorname{Spec} \prod_{\mathbb{Z}} k = \mathbb{Z}$  is a subgroup if and only if  $\operatorname{Spec} \mathbf{A}^* = n \cdot \mathbb{Z}$ . Dually, the quotient algebraic groups of  $G_m$  are the epimorphisms  $G_m \rightarrow G_m$ ,  $t \mapsto t^n$ . Hence, the subgroups of  $G_m$  are the  $\mu_n = \{\alpha \in G_m : \alpha^n = 1\} = \operatorname{Spec} k[x]/(x^n - 1) = (\mathbb{Z}/(n))^*$ .

Likewise,  $(G_m \times .n. \times G_m)^* = \mathbb{Z} \times .n. \times \mathbb{Z}$ . By the theory of abelian groups (or  $\mathbb{Z}$ -modules), the subgroups of  $\mathbb{Z} \times .n. \times \mathbb{Z}$  are isomorphic to  $\mathbb{Z}^r$  through an injective morphism  $\phi$  (let us say that its matrix is  $(n_{ij})$ ),

$$\mathbb{Z} \times .n. \times \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \times .n. \times \mathbb{Z}$$

whose cokernel is isomorphic to  $\mathbb{Z}^{n-r} \times \mathbb{Z}/(n_1) \times \dots \times \mathbb{Z}/(n_r)$ . Dually, the subgroups of  $G_m \times .n. \times G_m$  are isomorphic to  $G_m^{n-r} \times \mu_{n_1} \times \dots \times \mu_{n_r}$ , that is isomorphic to the kernel of the epimorphism

$$G_m^n \rightarrow G_m^r, (t_1, \dots, t_n) \mapsto (t_1^{n_{11}} \dots t_n^{n_{n1}}, \dots, t_1^{n_{1r}} \dots t_n^{n_{nr}}).$$

**Proposition 3.8.** [B, Ch. III, 8.12] The category of diagonalizable algebraic groups is anti-equivalent to the category of  $\mathbb{Z}$ -modules of finite type (or finitely-generated abelian groups).

**Example 3.9 (Affine toric varieties).** Let  $M$  be a set with structure of abelian (multiplicative) monoid. The constant functor  $\mathcal{M} = \operatorname{Spec} \prod_M k$  is a functor of abelian monoids. The dual functor is the abelian  $k$ -monoid  $\mathcal{M}^* = \operatorname{Spec} \bigoplus_M k = \operatorname{Spec} k[M]$ .

We will say that an abelian monoid  $M$  is classic if it is finitely generated and its associated group  $G$  is torsion-free and the natural morphism  $M \rightarrow G$  is injective. It is easy to prove that  $M$  is classic if and only if  $k[M]$  is an integral finitely generated  $k$ -algebra.

**Theorem.** The category of abelian monoids (respectively finitely generated, classic) is anti-equivalent to the category of affine  $k$ -schemes of semisimple abelian monoids (respectively algebraic, integral algebraic).

As  $G = \mathbb{Z}^n$ , the morphism  $M \rightarrow G$  induces a morphism  $G_m^n \rightarrow \mathcal{M}^*$ . In particular,  $G_m^n$  operates on  $\mathcal{M}^*$ . Furthermore, as  $k[G]$  is the localization of  $k[M]$  by the algebraically closed system  $M$ , the morphism  $G_m^n \rightarrow \mathcal{M}^*$  is an open injection. We will say that an integral affine algebraic variety on which the torus operates with a dense orbit is an affine toric variety. It is easy to prove that there exists a one-to-one correspondence between affine toric varieties with a fixed point whose orbit is transitive and dense, and classic monoids.

**Example 3.10.** Let  $G_a = \operatorname{Spec} k[x]$  be the additive group. Let us compute  $G_a^*$ . Assume  $\operatorname{car} k = 0$ . Let  $k[[z]]$  be the  $k$ -algebra scheme defined by  $k[[z]](B) := B[[z]]$ . Let  $\{\omega_0, \dots, \omega_n, \dots\} \subset k[x]^*$  be the dual basis of  $\{1, x, \dots, x^n, \dots\} \subset k[x]$ . Let  $\phi : k[x]^* \rightarrow k[[z]]$  be the linear isomorphism defined by  $\phi(\omega_i) = \frac{z^i}{i!}$ . The composition of the natural morphism  $G_a \rightarrow k[x]^*$  with  $\phi$  is equal to the morphism  $G_a \rightarrow k[[z]]$ ,  $\alpha \mapsto e^{\alpha z}$ , which is a morphism of monoids. Therefore,  $\phi$  is an isomorphism of functors of algebras.



$$\begin{aligned} G_a^*(B) &= \text{Hom}_{k\text{-alg}}(\mathbf{k}[\mathbf{x}]^*, \mathbf{B}) = \text{Hom}_{k\text{-alg}}(\mathbf{k}[[\mathbf{z}]], \mathbf{B}) \\ &= \varinjlim_n \text{Hom}_{k\text{-alg}}(k[\mathbf{z}]/(z^n), B) = \text{rad } B. \end{aligned}$$

Let us follow the notation  $G_a^* = \text{rad } \mathbf{k}$ . Explicitly, we assign to  $n \in \text{rad } \mathbf{k}$  the morphism  $G_a \rightarrow G_m$ ,  $\alpha \mapsto e^{\alpha \cdot n}$ . Finally,  $(\text{rad } \mathbf{k})^* = G_a$ ; explicitly,  $\alpha \in G_a$  defines the morphism  $\text{rad } \mathbf{k} \rightarrow G_m$ ,  $u \mapsto e^{\alpha \cdot u}$ .

**Proposition 3.11.** *Let  $G = \text{Spec } A$  be an abelian  $k$ -monoid. Let  $p \in G^*(k)$  and let us regard  $k$  as a  $G$ -module via  $p$ . Then*

$$T_p G^* := \text{Der}_k(\mathbf{A}^*, \mathbf{k}) = \text{Der}(G, \mathbf{k}).$$

*Let  $q \in G(k)$  and let us regard  $k$  as a  $G^*$ -module via  $q$ . Then*

$$T_q G = \text{Der}_k(\mathbf{A}, \mathbf{k}) = \text{Der}(G^*, \mathbf{k}).$$

*Proof.* It is a consequence from Proposition 2.10, where  $\mathcal{V} = \mathbf{k}$ .  $\square$

Let  $\mathbf{I}_p^* := \ker p$ , then  $\text{Hom}_k(\mathbf{I}_p^*/\mathbf{I}_p^{*2}, \mathbf{k}) = \text{Der}_k(\mathbf{A}^*, \mathbf{k}) = \text{Der}(G, \mathbf{k})$ .

#### 4. STRUCTURE OF COMMUTATIVE $k$ -GROUPS IN CHARACTERISTIC ZERO

For simplicity we assume that  $k$  is an algebraically closed field. Let  $\mathbf{A}^*$  be a  $k$ -scheme of commutative algebras and  $\mathbf{M}^* = \prod_J \mathbf{k}$  the maximal semisimple quotient of  $\mathbf{A}^*$ . The maximal spectrum of a scheme of algebras is defined to be the set of its maximal bilateral ideal schemes and it is the same as the set of isomorphism classes of simple  $\mathbf{A}^*$ -modules (see [A1, 6.7]). Then,  $\text{Spec}_{\max} \mathbf{A}^* = \text{Spec}_{\max} \mathbf{M}^* = J$ .  $\mathbf{A}^*$  is the inverse limit of its cokernels of finite dimension. As every finite commutative  $k$ -algebra is a direct product of local  $k$ -algebras, it is easy to prove that  $\mathbf{A}^* = \prod_{j \in J} \mathbf{A}_j^*$ , where the maximal semisimple quotient of  $\mathbf{A}_j^*$  is  $\mathbf{k}$ . Then, there exists an only section of algebra  $k$ -schemes  $s : \mathbf{M}^* \rightarrow \mathbf{A}^*$  of the epimorphism  $\mathbf{A}^* \rightarrow \mathbf{M}^*$ .

The maximal semisimple quotient of  $\mathbf{A}^* \bar{\otimes} \mathbf{A}^* = (\mathbf{A} \otimes \mathbf{A})^*$  is  $\mathbf{M}^* \bar{\otimes} \mathbf{M}^* = (\mathbf{M} \otimes \mathbf{M})^*$  ([A1, 6.23]). If  $\text{Spec } \mathbf{A}^*$  is also a functor of abelian groups, then  $\text{Spec } \mathbf{M}^*$ , which is the constant functor  $J$ , is a functor of abelian subgroups. Furthermore, the morphism  $s^* : \text{Spec } \mathbf{A}^* \rightarrow \text{Spec } \mathbf{M}^*$  induced by  $s$  is a morphism of functors of groups. Therefore, if  $1 \in J$  is the identity element, we have the following decomposition of group functors

$$\text{Spec } \mathbf{A}^* = \text{Spec } \mathbf{M}^* \times \text{Spec } \mathbf{A}_1^*.$$

In particular,  $G = \text{Spec } A$ , which is the dual of  $G^* = \text{Spec } \mathbf{A}^*$ , is a direct product of a diagonalizable group (every linear representation is a direct sum of simple modules of rank 1) and a unipotent group (every linear representation has got a filtration of invariant factors).

Given a  $k$ -vector space scheme  $\mathbf{E}^*$ , let us denote by  $\bar{S}^n \mathbf{E}^*$  the representant of the functor

$$F(\mathbf{V}^*) = \text{Sim}_k(\mathbf{E}^* \times \dots \times \mathbf{E}^*, \mathbf{V}^*).$$

$\bar{S}^n \mathbf{E}^*$  is the quotient of  $\mathbf{E}^* \bar{\otimes} \dots \bar{\otimes} \mathbf{E}^* = (\mathbf{E} \otimes \dots \otimes \mathbf{E})^*$  by the minimum  $k$ -vector space subscheme that contains elements of the following type

$$\dots \otimes w \otimes \dots \otimes w' \otimes \dots - \dots \otimes w' \otimes \dots \otimes w \otimes \dots$$

$\bar{S}^n \mathbf{E}^*$  coincides with the closure of  $k$ -vector space schemes of  $S^n \mathbf{E}^*$  and it coincides with  $((\mathbf{E} \otimes \dots \otimes \mathbf{E})^{S_n})^*$ . In characteristic zero,  $(\mathbf{E} \otimes \dots \otimes \mathbf{E})^{S_n} = S^n E$ , then  $\bar{S}^n \mathbf{E}^* = (S^n \mathbf{E})^*$ .

If  $E = \oplus_X k$ , then  $\mathbf{E}^* = \prod_X \mathbf{k}$  and  $\bar{S}^n \mathbf{E}^* = \prod_{S^n X} \mathbf{k}$ .

We will denote  $k[[\mathbf{E}^*]] := \prod_{n \in \mathbb{N}} \bar{S}^n \mathbf{E}^*$ . If  $\text{car } k = 0$  and  $G = \mathbf{V}^*$ , then  $G^* = \text{Spec } k[[\mathbf{V}^*]]$ . Specifically, the natural morphism  $\mathbf{V}^* \rightarrow k[[\mathbf{V}^*]]$  assigns  $w \in \mathbf{V}^*$  to  $e^w$ .

**Definition 4.1.** *The kernel of the morphism  $\mathbf{A}^* \rightarrow \mathbf{M}^*$  (the morphism of a scheme of algebras onto its maximal semisimple quotient) is said to be the radical of  $\mathbf{A}^*$ . We will say that  $\mathbf{A}^*$  is local if  $\mathbf{M}^* = \mathbf{k}$ .*

**Lemma 4.2.** *Let  $f : \mathbf{A}^* \rightarrow \mathbf{B}^*$  be a morphism between local  $k$ -algebra schemes and let  $\mathbf{I}_A^*$  and  $\mathbf{I}_B^*$  be the radical ideals of  $\mathbf{A}^*$  and  $\mathbf{B}^*$ , respectively. If the induced morphism  $\mathbf{I}_A^*/\mathbf{I}_A^{*2} \rightarrow \mathbf{I}_B^*/\mathbf{I}_B^{*2}$  is surjective, then  $f$  is a surjective morphism. If  $\mathbf{B}^* = k[[\mathbf{E}^*]]$  and the morphism  $\mathbf{I}_A^*/\mathbf{I}_A^{*2} \rightarrow \mathbf{I}_B^*/\mathbf{I}_B^{*2}$  is injective, then  $f$  is an injective morphism.*

*Proof.* By abuse of notation, we have regarded  $\mathbf{I}^{*n}$  as the closure of vector space schemes of  $\mathbf{I}^{*n}$  in  $\mathbf{A}^*$  (as we did in [A1]). It is convenient to denote by  $(\mathbf{I}^{*n})'$  this closure in this paragraph. The natural morphism  $S^n \mathbf{I}^*/\mathbf{I}^{*2} \rightarrow \mathbf{I}^{*n}/\mathbf{I}^{*n+1}$  is surjective and the closure of  $k$ -vector space schemes of the image of  $\mathbf{I}^{*n}/\mathbf{I}^{*n+1}$  in  $\mathbf{A}^*/(\mathbf{I}^{*n+1})'$  is  $(\mathbf{I}^{*n})'/(\mathbf{I}^{*n+1})'$ . Therefore, the morphism  $\bar{S}^n(\mathbf{I}^*/\mathbf{I}^{*2}) \rightarrow (\mathbf{I}^{*n})'/(\mathbf{I}^{*n+1})'$  is surjective.

If  $\mathbf{I}_A^*/\mathbf{I}_A^{*2} \rightarrow \mathbf{I}_B^*/\mathbf{I}_B^{*2}$  is a surjective morphism, then the morphisms  $\bar{S}^n(\mathbf{I}_A^*/\mathbf{I}_A^{*2}) \rightarrow \bar{S}^n(\mathbf{I}_B^*/\mathbf{I}_B^{*2})$  are surjective and from the commutative diagram

$$\begin{array}{ccc} \bar{S}^n(\mathbf{I}_A^*/\mathbf{I}_A^{*2}) & \xrightarrow{\text{epi}} & \bar{S}^n(\mathbf{I}_B^*/\mathbf{I}_B^{*2}) \\ \downarrow \text{epi} & & \downarrow \text{epi} \\ \mathbf{I}_A^{*n}/\mathbf{I}_A^{*n+1} & \longrightarrow & \mathbf{I}_B^{*n}/\mathbf{I}_B^{*n+1} \end{array}$$

it is deduced that the morphisms  $\mathbf{I}_A^{*n}/\mathbf{I}_A^{*n+1} \rightarrow \mathbf{I}_B^{*n}/\mathbf{I}_B^{*n+1}$  are surjective. Then the morphisms  $\mathbf{A}^*/\mathbf{I}_A^{*n+1} \rightarrow \mathbf{B}^*/\mathbf{I}_B^{*n+1}$  are surjective. Taking inverse limits (which is the dual of a direct limit of injections) we get the epimorphism  $f : \mathbf{A}^* \rightarrow \mathbf{B}^*$  ([A1, 6.17 (2)]).

If  $\mathbf{I}_A^*/\mathbf{I}_A^{*2} \rightarrow \mathbf{I}_B^*/\mathbf{I}_B^{*2}$  is an injective morphism, then the morphisms  $\bar{S}^n(\mathbf{I}_A^*/\mathbf{I}_A^{*2}) \rightarrow \bar{S}^n(\mathbf{I}_B^*/\mathbf{I}_B^{*2})$  are injective and from the commutative diagram

$$\begin{array}{ccc} \bar{S}^n(\mathbf{I}_A^*/\mathbf{I}_A^{*2}) & \hookrightarrow & \bar{S}^n(\mathbf{I}_B^*/\mathbf{I}_B^{*2}) \\ \downarrow \text{epi} & & \parallel \\ \mathbf{I}_A^{*n}/\mathbf{I}_A^{*n+1} & \longrightarrow & \mathbf{I}_B^{*n}/\mathbf{I}_B^{*n+1} \end{array}$$

it is deduced that the morphisms  $\mathbf{I}_A^{*n}/\mathbf{I}_A^{*n+1} \rightarrow \mathbf{I}_B^{*n}/\mathbf{I}_B^{*n+1}$  are injective. Let  $0 \neq a \in \mathbf{A}^*$  and let  $n$  be such that  $a \in \mathbf{I}_A^{*n}$  and  $a \notin \mathbf{I}_A^{*n+1}$  (see [A1, 6.17 (2)]). Then  $f(a) \neq 0$ , since  $0 \neq \bar{a} \in \mathbf{I}_A^{*n}/\mathbf{I}_A^{*n+1}$  and  $0 \neq \bar{f(a)} \in \mathbf{I}_B^{*n}/\mathbf{I}_B^{*n+1}$ .  $\square$

**Theorem 4.3.** *Let  $G = \text{Spec } A$  be an abelian  $k$ -group and let  $k$  be an algebraically closed field of characteristic zero. Then, there exists an isomorphism of  $k$ -groups*

$$G = \left( \prod_X G_a \right) \times D$$

where  $D$  is a diagonalizable group.

*Proof.* We must prove that if  $G$  is unipotent, then it is isomorphic to a direct product of additive groups.

Thus, we have that  $\mathbf{A}^*$  is local. Let  $\mathbf{I}^*$  be its radical. The morphism  $i : G \rightarrow 1 + \mathbf{I}^*/\mathbf{I}^{*2}$ ,  $i(g) = 1 + \overline{g - 1}$  is a morphism of  $k$ -groups. Let us prove that it is an isomorphism. Consider  $\mathbf{k}$  as the trivial  $G$ -module. Then,  $E := \text{Der}(G, \mathbf{k}) = \text{Hom}_{\text{lin}}(G, \mathbf{k})$ . The ring of functions of  $1 + \mathbf{I}^*/\mathbf{I}^{*2} = \mathbf{I}^*/\mathbf{I}^{*2}$  is  $\oplus_n S^n E$ . The morphism between the rings induced by  $i$  is the natural morphism  $\oplus_n S^n E \rightarrow A$ . Dually, we have the

morphism  $\mathbf{A}^* \rightarrow k[[\mathbf{E}^*]] =: \mathbf{B}^*$ . The induced morphism  $\mathbf{I}^*/\mathbf{I}^{*2} \rightarrow \mathbf{I}_B^*/\mathbf{I}_B^{*2} = \mathbf{E}^*$  is the identity morphism. By the previous lemma,  $\mathbf{A}^* = k[[\mathbf{E}^*]]$  and  $G = \mathbf{I}^*/\mathbf{I}^{*2}$ .  $\square$

## 5. INVERSION FORMULA

Let  $\mathbf{B} \subseteq \prod_J \mathbf{k}$  be a quasi-coherent  $k$ -algebra (with or without unit). We define  $\text{Spec } \mathbf{B}$  to be the functor over the category of commutative algebras (with unit)

$$(\text{Spec } \mathbf{B})(C) := \{f \in \text{Hom}_{k\text{-alg}}(B, C) : (\text{Im } f) = C\}.$$

If  $k$  is a field it is easily seen that  $\text{Spec } \mathbf{B} = \text{Spec } \prod_J \mathbf{k}$  if and only if  $\mathbf{B} = \oplus_J \mathbf{k}$ .

**Definition 5.1.** *We will say that  $\oplus_J k$  is the quasi-coherent ring of functions of  $\text{Spec } \prod_J \mathbf{k}$ .*

As well,  $\oplus_J \mathbf{k}$  is characterized by being the maximal quasi-coherent  $\prod_J \mathbf{k}$ -submodule of  $\prod_J \mathbf{k}$ .

Let  $G^* = \text{Spec } \prod_J \mathbf{k}$  be a functor of discrete groups.  $G^*$  operates on  $G^*$  by translations on the left, then it operates on  $\prod_J \mathbf{k}$ , then on  $\oplus_J k$ . The mapping  $w_{G^*} : \oplus_J k \rightarrow k$ ,  $w_{G^*}((\lambda_i)) = \sum_i \lambda_i$  is the only  $G^*$ -invariant linear form up to a multiplicative factor. We will say that the linear mapping

$$\phi_{G^*} : \oplus_J \mathbf{k} \rightarrow \left(\prod_J \mathbf{k}\right)^* \hookrightarrow (\oplus_J \mathbf{k})^*, a \mapsto w_{G^*}(a \cdot -)$$

is the transposed Fourier transform of  $G^*$ .

$G = \text{Spec } A$  is a semisimple  $k$ -group if and only if there exists a (unique) linear mapping  $w_G : A \rightarrow k$ ,  $G$ -invariant and such that  $w_G(1) = 1$  (see [A2, 2.10]). The morphism  $\phi_G : A \rightarrow A^*$ ,  $\phi_G(a) := w_G(a^* \cdot -)$  (where  $*$  :  $A \rightarrow A$  is the morphism between rings induced by the morphism  $G \rightarrow G, g \mapsto g^{-1}$ ) is called the Fourier transform of  $G$  (see [A3]). If  $G$  is also commutative, then  $A^* = \prod_J k$  and  $\text{Im } \phi_G = \oplus_J k =: A'$ . Let  $e : A \rightarrow k$  the identity element of  $G$  (that is, the unit of  $A^*$ ). The diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi_G} & A' \\ & \searrow e & \swarrow w_{G^*} \\ & k & \end{array}$$

is commutative (see [A3], section 2).

**Theorem 5.2.** *The transposed Fourier transform of the dual group of a semisimple abelian  $k$ -group is the inverse of the Fourier transform of the semisimple abelian group.*

*Proof.*  $\phi_{G^*}(\phi_G(a))(w) = w_{G^*}(\phi_G(a) \cdot w) = w_{G^*}(\phi_G(a \cdot w)) = (a \cdot w)(e) = a(w)$  for all  $a \in A$  and  $w \in A^*$ .  $\square$

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